Inclusion of generalized Bessel functions in the Janowski class

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Abstract. Sufficient conditions on A, B, p, b and c are determined that will ensure the generalized Bessel functions $u_{p,b,c}$ satisfies the subordination $u_{p,b,c}(z) \prec (1+Az)/(1+Bz)$. In particular this gives conditions for $(-4\kappa/c)(u_{p,b,c}(z)-1)$, $c \neq 0$ to be close-to-convex. Also, conditions for which $u_{p,b,c}(z)$ to be Janowski convex, and $zu_{p,b,c}(z)$ to be Janowski starlike in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ are obtained.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f defined in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ normalized by the conditions $f(0) = 0 = f'(0) - 1$. If f and g are analytic in $\mathbb{D}$, then f is subordinate to g, written $f(z) \prec g(z)$, if there is an analytic self-map w of $\mathbb{D}$ satisfying $w(0) = 0$ and $f = g \circ w$. For $-1 \leq B < A \leq 1$, let $\mathcal{P}[A, B]$ be the class consisting of normalized analytic functions $p(z) = 1 + c_1 z + \cdots$ in $\mathbb{D}$ satisfying

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

For instance, if $0 \leq \beta < 1$, then $\mathcal{P}[1 - 2\beta, -1]$ is the class of functions $p(z) = 1 + c_1 z + \cdots$ satisfying $\operatorname{Re} p(z) > \beta$ in $\mathbb{D}$.

The class $\mathcal{S}^*[A, B]$ of Janowski starlike functions [8] consists of $f \in \mathcal{A}$ satisfying

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}[A, B].$$

For $0 \leq \beta < 1$, $\mathcal{S}^*[1 - 2\beta, -1] := \mathcal{S}^*(\beta)$ is the usual class of starlike functions of order β; $\mathcal{S}^*[1 - \beta, 0] := \mathcal{S}^*_\beta = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < 1 - \beta\}$, and $\mathcal{S}^*[\beta, -\beta] := \mathcal{S}^*[\beta] = \{f \in \mathcal{A} : |zf'(z)/f(z) - 1| < \beta|zf'(z)/f(z) + 1|\}$. These classes have been studied, for example, in [1, 2]. A function $f \in \mathcal{A}$ is said to be close-to-convex of order β [7, 12] if $\operatorname{Re}(zf'(z)/g(z)) > \beta$ for some $g \in \mathcal{S}^* := \mathcal{S}^*(0)$.

This article studies the generalized Beesel function $u_p(z) = u_{p,b,c}(z)$ given by the power series

$$u_p(z) = {}_0F_1(\kappa, \tfrac{-c}{4}z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{4^k (\kappa)_k} \frac{z^k}{k!}, \tag{1}$$

where $\kappa = p + (b+1)/2 \neq 0, -1, -2, -3 \cdots$. The function $u_p(z)$ is analytic in $\mathbb{D}$ and solution of the differential equation

$$4z^2 u''(z) + 4\kappa z u'(z) + czu(z) = 0, \tag{2}$$

if b,p,c in $\mathbb{C}$,such that $\kappa = p + (b+1)/2 \neq 0, -1, -2, -3 \cdots$ and $z \in \mathbb{D}$. This normalized and generalized Bessel function of the first kind of order p, also satisfy the following recurrence relation

$$4\kappa u_p'(z) = -cu_{p+1}(z), \tag{3}$$

which is an useful tool to study several geometric properties of u_p. There has been several works [3, 4, 15, 16, 5, 6] studying geometric properties of the function $u_p(z)$, such as on its close-to-convexity, starlikeness, and convexity, radius of starlikeness and convexity.

In Section 2 of this paper, sufficient conditions on A, B, c, κ are determined that will ensure u_p satisfies the subordination $u_p(z) \prec (1+Az)/(1+Bz)$. It is to be understood that a computationally-intensive methodology with shrewd manipulations is required to obtain the results in this general framework. The benefits of such general results are that by judicious choices of the parameters A and B, they give rise to several interesting applications, which include extending the results of previous works. Using this subordination result, sufficient conditions are obtained for $(-4\kappa/c)u'(z) \in \mathcal{P}[A, B]$, which next readily gives conditions for $(-4\kappa/c)(u_p(z) - 1)$ to be close-to-convex. Section 3 gives emphasis to the investigation of $u_p(z)$ to be Janowski convex as well as of $zu_p(z)$ to be Janowski starlike.

The following lemma is needed in the sequel.

Lemma 1.1. [11, 12] *Let* $\Omega \subset \mathbb{C}$, *and* $\Psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ *satisfy*

$$\Psi(i\rho, \sigma; z) \notin \Omega$$

whenever $z \in \mathbb{D}$, ρ *real,* $\sigma \leq -(1+\rho^2)/2$. *If* p *is analytic in* $\mathbb{D}$ *with* $p(0) = 1$, *and* $\Psi(p(z), zp'(z); z) \in \Omega$ *for* $z \in \mathbb{D}$, *then* $\operatorname{Re} p(z) > 0$ *in* $\mathbb{D}$.

In the case $\Psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$, then the condition in Lemma 1.1 generalized to

$$\Psi(i\rho, \sigma, \mu + i\nu; z) \notin \Omega$$

ρ real, $\sigma + \mu \leq 0$ and $\sigma \leq -(1+\rho^2)/2$.

2. Close-to-convexity of the Bessel function

In this section, one main result on the close-to-convexity of the generalized Bessel function with several consequences are discussed in details.

Theorem 2.1. *Let* $-1 \leq B \leq 3-2\sqrt{2} \approx 0.171573$. *Suppose* $B < A \leq 1$, *and* c, $\kappa \in \mathbb{R}$ *satisfy*

$$\kappa - 1 \geq \max\left\{0, \tfrac{(1+B)(1+A)}{4(A-B)}\,|c|\right\}. \tag{4}$$

Further let A, B, κ *and* c *satisfy either the inequality*

$$(\kappa-1)^2 + \tfrac{(\kappa-1)(1+B)}{(1-B)} - \left|\tfrac{(\kappa-1)(A+B)}{2(A-B)}c + \tfrac{(1+B)^2(1+A)}{4(1-B)(A-B)}c\right| \geq \tfrac{(1-A^2)(1-B^2)}{16(A-B)^2}c^2 \tag{5}$$

whenever

$$\left|2(\kappa-1)(1-B)(A+B)c + (1+B)^2(1+A)c\right| \geq \tfrac{1}{2}(A-B)(1-B)c^2, \tag{6}$$

or the inequality

$$\begin{aligned}\left((\kappa-1)\tfrac{(A+B)}{2(A-B)}c + \tfrac{(1+B)^2((1+A)}{4(1-B)(A-B)}c\right)^2 \\ \leq \tfrac{c^2}{4}\left((\kappa-1)^2 + (\kappa-1)\tfrac{(1+B)}{1-B} - \tfrac{(1-AB)^2}{16(A-B)^2}c^2\right)\end{aligned} \tag{7}$$

whenever

$$\left|2(\kappa-1)(1-B)(A+B)c + (1+B)^2(1+A)c\right| < \tfrac{1}{2}(A-B)(1-B)c^2. \tag{8}$$

If $(1+B)u_p(z) \neq (1+A)$, *then* $u_p(z) \in \mathcal{P}[A,B]$.

Proof. Define the analytic function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = -\tfrac{(1-A)-(1-B)u_p(z)}{(1+A)-(1+B)u_p(z)},$$

Then, a computation yields

$$u_p(z) = \tfrac{(1-A)+(1+A)p(z)}{(1-B)+(1+B)p(z)}, \tag{9}$$

$$u_p(z) = \tfrac{2(A-B)p'(z)}{((1-B)+(1+B)p(z))^2}, \tag{10}$$

and

$$u_p''(z) = \tfrac{2(A-B)((1-B)+(1+B)p(z))p''(z) - 4(1+B)(A-B)p'^2(z)}{((1-B)+(1+B)p(z))^3}. \tag{11}$$

Thus, using the identities (9)–(11), the Bessel differential equation (2) can be rewrite as

$$\begin{aligned}z^2p''(z) - \tfrac{2(1+B)}{(1-B)+(1+B)p(z)}(zp'(z))^2 + \kappa zp'(z) \\ + \tfrac{((1-B)+(1+B)p(z))((1-A)+(1+A)p(z))}{8(A-B)}cz = 0.\end{aligned} \tag{12}$$

Assume $\Omega = \{0\}$, and define $\Psi(r,s,t;z)$ by

$$\Psi(r,s,t;z) := t - \tfrac{2(1+B)}{(1-B)+(1+B)r}s^2 + \kappa s + \tfrac{((1-B)+(1+B)r)((1-A)+(1+A)r)}{8(A-B)}cz. \tag{13}$$

It follows from (12) that $\Psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$. To ensure $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$, from Lemma 1.1, it is enough to establish $\operatorname{Re}\Psi(i\rho, \sigma, \mu + i\nu; z) < 0$ in $\mathbb{D}$ for any real ρ, $\sigma \leq -(1+\rho^2)/2$, and $\sigma + \mu \leq 0$.

With $z = x + iy \in \mathbb{D}$ in (13), a computation yields

$$\operatorname{Re}\Psi(i\rho, \sigma, \mu + i\nu; z) = \mu - \tfrac{2(1-B^2)}{(1-B)^2+(1+B)^2\rho^2}\sigma^2 + \kappa\sigma - \tfrac{\rho(1-AB)}{4(A-B)}cy$$

$$+\tfrac{(1-B)(1-A)-(1+B)(1+A)\rho^2}{8(A-B)}cx. \tag{14}$$

Since $\sigma \leq -(1+\rho^2)/2$, and $B \in [-1, 3-2\sqrt{2}]$,

$$\tfrac{2(1-B^2)}{(1-B)^2+(1+B)^2\rho^2}\sigma^2 \geq \tfrac{2(1-B^2)}{(1-B)^2+(1+B)^2\rho^2}\tfrac{(1+\rho^2)^2}{4} \geq \frac{1+B}{2(1-B)}.$$

Thus

$$\begin{aligned}\operatorname{Re}\Psi(i\rho,\sigma,\mu+i\nu;z) &\leq (\kappa-1)\sigma - \tfrac{1+B}{2(1-B)} - \tfrac{\rho(1-AB)}{4(A-B)}cy \\ &\qquad + \tfrac{(1-B)(1-A)-(1+B)(1+A)\rho^2}{8(A-B)}cx \\ &\leq -\tfrac{1}{2}(\kappa-1)(1+\rho^2) - \tfrac{1+B}{2(1-B)} - \tfrac{\rho(1-AB)}{4(A-B)}cy \\ &\qquad + \tfrac{(1-B)(1-A)-(1+B)(1+A)\rho^2}{8(A-B)}cx \\ &= p_1\rho^2 + q_1\rho + r_1 := Q(\rho),\end{aligned}$$

where

$$\begin{aligned} p_1 &= -\tfrac{1}{2}(\kappa-1) - \tfrac{(1+B)(1+A)cx}{8(A-B)}, \\ q_1 &= -\tfrac{1-AB}{4(A-B)}cy, \\ r_1 &= -\tfrac{1}{2}(\kappa-1) + \tfrac{(1-B)(1-A)}{8(A-B)}cx - \tfrac{1+B}{2(1-B)}.\end{aligned}$$

Condition (4) shows that

$$\begin{aligned} p_1 &= -\tfrac{1}{2}(\kappa-1) - \tfrac{(1+B)(1+A)cx}{8(A-B)} \\ &< -\tfrac{1}{2}\left((\kappa-1) - \tfrac{(1+B)(1+A)}{4(A-B)}\,|c|\right) < 0.\end{aligned}$$

Since $\max\limits_{\rho\in\mathbb{R}}\{p_1\rho^2+q_1\rho+r_1\} = (4p_1r_1-q_1^2)/(4p_1)$ for $p_1<0$, it is clear that $Q(\rho)<0$ when

$$\begin{aligned}\tfrac{(1-AB)^2}{16(A-B)^2}c^2y^2 < 4\left(-\tfrac{1}{2}(\kappa-1) - \tfrac{(1+B)(1+A)}{8(A-B)}cx\right) \times \\ \left(-\tfrac{1}{2}(\kappa-1) + \tfrac{(1-B)(1-A)}{8(A-B)}cx - \tfrac{1+B}{2(1-B)}\right),\end{aligned}$$

with $|x|,|y|<1$. As $y^2<1-x^2$, the above condition holds whenever

$$\begin{aligned}&\tfrac{(1-AB)^2c^2}{16(A-B)^2}(1-x^2) \\ &\leq \left((\kappa-1) + \tfrac{(1+B)(1+A)cx}{4(A-B)}\right)\left((\kappa-1) - \tfrac{(1-B)(1-A)}{4(A-B)}cx + \tfrac{1+B}{1-B}\right),\end{aligned}$$

that is, when

$$\begin{aligned}\tfrac{c^2}{16}x^2 + \left((\kappa-1)\tfrac{(A+B)}{2(A-B)}c + \tfrac{(1+B)^2(1+A)}{4(1-B)(A-B)}c\right)x \\ + (\kappa-1)^2 + (\kappa-1)\tfrac{1+B}{1-B} - \tfrac{(1-AB)^2}{16(A-B)^2}c^2 \geq 0.\end{aligned} \tag{15}$$

To establish inequality (15), consider the polynomial R given by

$$R(x) := mx^2 + nx + r, \quad |x|<1,$$

where

$$m := \tfrac{c^2}{16}, \qquad n := (\kappa-1)\tfrac{(A+B)}{2(A-B)}c + \tfrac{(1+B)^2(1+A)}{4(1-B)(A-B)}c$$
$$r := (\kappa-1)^2 + (\kappa-1)\tfrac{1+B}{1-B} - \tfrac{(1-AB)^2}{16(A-B)^2}c^2.$$

The constraint (6) yields $|n| \geq 2|m|$, and thus $R(x) \geq m + r - |n|$. Now inequality (5) readily implies that

$$\begin{aligned} R(x) &\geq m + r - |n| \\ &= \tfrac{c^2}{16} + (\kappa-1)^2 + (\kappa-1)\tfrac{1+B}{1-B} - \tfrac{(1-AB)^2}{16(A-B)^2}c^2 \\ &\quad - \left|(\kappa-1)\tfrac{(A+B)}{2(A-B)}c + \tfrac{(1+B)^2(1+A)}{4(1-B)(A-B)}c\right| \\ &= (\kappa-1)^2 + (\kappa-1)\tfrac{(1+B)}{1-B} - \left|(\kappa-1)\tfrac{(A+B)}{2(A-B)}c + \tfrac{(1+B)^2(1+A)}{4(1-B)(A-B)}c\right| \\ &\quad - \tfrac{(1-A^2)(1-B^2)}{16(A-B)^2}c^2 \\ &\geq 0. \end{aligned}$$

Now considers the case of the constraint (8), which is equivalent to $|n| < 2m$. Then the minimum of R occurs at $x = -n/(2m)$, and (7) yields

$$R(x) \geq \tfrac{4mr-n^2}{4m} \geq 0.$$

Evidently Ψ satisfies the hypothesis of Lemma 1.1, and thus Re $p(z) > 0$, that is,

$$-\tfrac{(1-A)-(1-B)u_p(z)}{(1+A)-(1+B)u_p(z)} \prec \tfrac{1+z}{1-z}.$$

Hence there exists an analytic self-map w of $\mathbb{D}$ with $w(0) = 0$ such that

$$-\tfrac{(1-A)-(1-B)u_p(z)}{(1+A)-(1+B)u_p(z)} = \tfrac{1+w(z)}{1-w(z)},$$

which implies that $u_p(z) \prec (1 + Az)/(1 + Bz)$. □

Theorem 2.1 gives rise to simple conditions on c and κ to ensure $u_p(z)$ maps $\mathbb{D}$ into a half-plane.

Corollary 2.1. *Let $c \leq 0$ and $2\kappa \geq 2 + c^2$. Then* $\operatorname{Re} u_p(z) > c/(c-1)$.

Proof. Choose $A = -(c+1)/(c-1)$, and $B = -1$ in Theorem 2.1. Then both the conditions (4) and (6) are equivalent to $\kappa \geq 1$ which clearly holds for $\kappa \geq 1 + c^2/2$. The proof will complete if the hypothesis (5) holds, i.e.,

$$(\kappa-1)^2 \geq \tfrac{1}{2}(\kappa-1)c^2. \tag{16}$$

Since $\kappa \geq 1 + c^2/2$, it follows that

$$(\kappa-1)^2 - \tfrac{1}{2}(\kappa-1)c^2 = (\kappa-1)\left(\kappa - 1 - \tfrac{c^2}{2}\right) \geq 0,$$

which establishes (16). □

Corollary 2.2. *Let c, κ be real such that*

$$\kappa \geq \begin{cases} 1, & c \leq 0 \\ 1+\frac{c}{2}, & c \geq 0. \end{cases}$$

Then $\operatorname{Re} u_p(z) > 1/2$.

Proof. Put $A=0$ and $B=-1$ in Theorem 2.1. The condition (4) reduces to $\kappa \geq 1$, which holds in all cases. It is sufficient to establish conditions (6) and (5), or equivalently,

$$4(\kappa-1)-c \geq 0, \tag{17}$$

and

$$(\kappa-1)^2 - \tfrac{1}{2}(\kappa-1)c \geq 0. \tag{18}$$

For the case when $c \leq 0$, both the inequality (17) and (18) hold as $\kappa \geq 1$.

Finally it is readily established for $c \geq 0$ and $\kappa - 1 \geq c/2$ that $4(\kappa - 1) - c \geq c \geq 0$, and $(\kappa-1)^2 - \frac{1}{2}(\kappa-1)c \geq (\kappa-1)(\kappa-1-\frac{c}{2}) \geq 0$. □

It is known that for $b=2$ and $c=\pm 1$, the generalized Bessel functions $u_{p,2,1}(z) = j_p(z)$ and $u_{p,2,-1}(z) = i_p(z)$ respectively gives the spherical Bessel and modified spherical Bessel functions. This specific choice of b and c, Corollary 2.2 yield $\operatorname{Re}(i_p(z)) > 1/2$ for $p \geq -1/2$, and $\operatorname{Re}(j_p(z)) > 1/2$, for $p \geq 0$. Since $i'_p(0) = 1/(4p+6)$ for $p \geq -1/2$, following inequalities can be obtain with the aid of results in [9].

Corollary 2.3. *For $p \geq -1/2$, the modified spherical Bessel functions i_p satisfy the following inequalities.*

$$|i_p(z)| \leq \frac{4p+6+|z|}{2(2p+3)(1-|z|^2)}, \tag{19}$$

$$\operatorname{Re}(i_p(z) \geq \frac{p+6+|z|}{4p+6+2|z|+2(2p+3)|z|^2}, \tag{20}$$

$$|i'_p(z)| \leq \frac{2\operatorname{Re}(i_p(z)-1}{2(1-|z|^2)} \times \frac{|z|^2+4(2p+3)|z|+1}{(2p+3)|z|^2+|z|+(2p+3)}. \tag{21}$$

Next theorem gives the sufficient condition for close-to-convexity when $B \geq 3-2\sqrt{2}$.

Theorem 2.2. *Let $3-2\sqrt{2} \leq B < A \leq 1$ and c, $\kappa \in \mathbb{R}$ satisfy*

$$\kappa - 1 \geq \max\left\{0, \tfrac{(1+B)(1+A)}{4(A-B)}\,|c|\right\}. \tag{22}$$

Suppose A, B, κ and c satisfy either the inequality

$$(\kappa-1)^2 + 16(\kappa-1)\tfrac{B(1-B)}{(1+B)^3} - \left|\tfrac{(\kappa-1)(A+B)}{2(A-B)}c + \tfrac{4B(1-B^2)(1+A)}{(1+B)^3(A-B)}c\right| \geq \tfrac{(1-A^2)(1-B^2)}{16(A-B)^2}c^2 \tag{23}$$

whenever

$$\left|(\kappa-1)(1+B)^3(A+B)c+8B(1-B^2)(1+A)c\right| \geq \tfrac{c^2}{4}(A-B)(1+B)^3, \tag{24}$$

or the inequality

$$\begin{aligned}&\left((\kappa-1)\tfrac{(A+B)}{2(A-B)}c+\tfrac{4B(1-B^2)((1+A)}{(1+B)^3(A-B)}c\right)^2\\ &\leq \tfrac{c^2}{4}\left((\kappa-1)^2+16(\kappa-1)\tfrac{B(1-B)}{(1+B)^3}-\tfrac{(1-AB)^2}{16(A-B)^2}c^2\right)\end{aligned} \tag{25}$$

whenever

$$\left|\left((\kappa-1)(A+B)(1+B)^3+8B(1-B^2)(1+A)\right)c\right| < \tfrac{c^2}{4}(A-B)(1+B)^3. \tag{26}$$

If $(1+B)u_p(z) \neq (1+A)$, *then* $u_p(z) \in \mathcal{P}[A,B]$.

Proof. First, proceed similar to the proof of Theorem 2.1 and derive the expression of $\operatorname{Re}\Psi(i\rho,\sigma,\mu+i\nu;z)$ as given in (14) . Now for $\sigma \leq -(1+\rho^2)/2$, $\rho\in\mathbb{R}$, and $B \geq 3-2\sqrt{2}$,

$$\tfrac{2(1-B^2)}{(1-B)^2+(1+B)^2\rho^2}\sigma^2 \geq \tfrac{2(1-B^2)}{(1-B)^2+(1+B)^2\rho^2}\tfrac{(1+\rho^2)^2}{4} \geq \frac{8B(1-B)}{(1+B)^3},$$

and then with $z=x+iy\in\mathbb{D}$, and $\mu+\sigma<0$, it follows that

$$\begin{aligned}&\operatorname{Re}\Psi(i\rho,\sigma,\mu+i\nu;z)\\ &\leq -\tfrac{1}{2}(\kappa-1)(1+\rho^2)-\tfrac{(1+B)(1+A)\rho^2}{8(A-B)}cx-\tfrac{\rho(1-AB)}{4(A-B)}cy\\ &\quad+\tfrac{(1-B)(1-A)}{8(A-B)}cx-\tfrac{8B(1-B)}{(1+B)^3}\\ &= p_2\rho^2+q_2\rho+r_2 := Q_1(\rho),\end{aligned}$$

where

$$\begin{aligned}p_2 &= -\tfrac{1}{2}(\kappa-1)-\tfrac{(1+B)(1+A)}{8(A-B)}cx,\\ q_2 &= -\tfrac{(1-AB)cy}{4(A-B)},\\ r_2 &= -\tfrac{1}{2}(\kappa-1)+\tfrac{(1-B)(1-A)}{8(A-B)}cx-\tfrac{8B(1-B)}{(1+B)^3}.\end{aligned}$$

Observe that the inequality (22) implies that $p_2<0$. Thus $Q_1(\rho)<0$ for all $\rho\in\mathbb{R}$ provided $q_2^2 \leq 4p_2r_2$, that is, for $|x|,|y|<1$,

$$\begin{aligned}&\tfrac{(1-AB)^2}{16(A-B)^2}c^2y^2\\ &\leq \left((\kappa-1)+\tfrac{(1+B)(1+A)}{4(A-B)}cx\right)\left((\kappa-1)-\tfrac{(1-B)(1-A)}{4(A-B)}cx+\tfrac{16B(1-B)}{(1+B)^3}\right).\end{aligned}$$

With $y^2<1-x^2$, it is enough to show for $|x|<1$,

$$\begin{aligned}&\tfrac{(1-AB)^2}{16(A-B)^2}c^2(1-x^2)\\ &\leq \left((\kappa-1)+\tfrac{(1+B)(1+A)}{4(A-B)}cx\right)\left((\kappa-1)-\tfrac{(1-B)(1-A)}{4(A-B)}cx+\tfrac{16B(1-B)}{(1+B)^3}\right),\end{aligned}$$

which is equivalent to

$$R_1(x) := m_1x^2 + n_1x + r_1 \geq 0, \tag{27}$$

where

$$\begin{aligned} m_1 &:= \tfrac{c^2}{16}, \\ n_1 &:= \left((\kappa-1)\left(\tfrac{c(A+B)}{2(A-B)}\right) + \tfrac{4B(1-B^2)(1+A)}{(A-B)(1+B)^3}c\right), \\ r_1 &:= (c-1)^2 + (c-1)\tfrac{16B(1-B)}{(1+B)^3} - \tfrac{a^2(1-AB)^2}{(A-B)^2}. \end{aligned}$$

If (24) holds, then $|n_1| \geq 2|m_1|$. Since R_1 is increasing, then $R_1(x) \geq m_1+r_1-|n_1|$, which is nonnegative from (23). On the other hand, if (26) holds, then $|n_1| < 2|m_1|$, $R_1(x) \geq (4m_1r_1 - n_1^2)/4m_1$, and (25) implies $R_1(x) \geq 0$. Either case establishes (27). □

Theorem 2.3. *Let $-1 \leq B \leq 3 - 2\sqrt{2} \approx 0.171573$. Suppose $B < A \leq 1$, $c, \kappa \in \mathbb{R}$ with $c \neq 0$ and satisfying*

$$\kappa \geq \max\left\{0, \tfrac{(1+B)(1+A)}{4(A-B)}\,|c|\right\}.$$

Further let A, B, κ and c satisfy either

$$\kappa^2 + \kappa\tfrac{1+B}{1-B} - \left|\kappa\tfrac{(A+B)}{2(A-B)}c + \tfrac{(1+B)^2(1+A)}{4(1-B)(A-B)}c\right| \geq \tfrac{(1-A^2)(1-B^2)}{16(A-B)^2}c^2$$

whenever

$$\left|2\kappa(1-B)(A+B)c + (1+B)^2(1+A)c)\right| \geq \tfrac{1}{2}(A-B)(1-B)c^2,$$

or the inequality

$$\left(\kappa\tfrac{(A+B)}{2(A-B)} + \tfrac{(1+B)^2(1+A)}{4(1-B)(A-B)}c\right)^2 \leq \tfrac{c^2}{4}\left(\kappa^2 + \tfrac{\kappa(1+B)}{1-B} - \tfrac{(1-AB)^2}{16(A-B)^2}c^2\right)$$

when

$$\left|2\kappa(1-B)(A+B)c + (1+B)^2(1+A)c\right| < \tfrac{1}{2}(A-B)(1-B)c^2.$$

If $(1+B)u_p(z) \neq (1+A)$, then $(-4\kappa/c)u_p'(z) \in \mathcal{P}[A,B]$.

Theorem 2.4. *Let $3 - 2\sqrt{2} < B < A \leq 1$. Suppose $c, \kappa \in \mathbb{R}$, $a \neq 0$, such that*

$$\kappa \geq \max\left\{0, \tfrac{(1+B)(1+A)}{4(A-B)}\,|c|\right\}.$$

Suppose A, B, κ and c satisfy either

$$\kappa^2 + 16\kappa\tfrac{B(1-B)}{(1+B)^3} - \left|\kappa\tfrac{(A+B)}{2(A-B)}c + \tfrac{4B(1-B^2)(1+A)}{(1+B)^3(A-B)}c\right| \geq \tfrac{(1-A^2)(1-B^2)}{16(A-B)^2}c^2$$

whenever

$$\left|\kappa(1+B)^3(A+B)c + 8B(1-B^2)(1+A)c)\right| \geq \tfrac{c^2}{4}(A-B)(1+B)^3,$$

or the inequality

$$\left(\kappa\tfrac{(A+B)}{2(A-B)}c + \tfrac{4B(1-B^2)(1+A)}{(1+B)^3(A-B)}\right)^2 \leq \tfrac{c^2}{4}\left(\kappa^2 + \tfrac{16\kappa(B(1-B)}{(1+B)^3} - \tfrac{(1-AB)^2}{16(A-B)^2}c^2\right)$$

when

$$\left|2\kappa(1+B)^3(A+B)c + 8B(1-B^2)(1+A)c\right| < \tfrac{c^2}{4}(A-B)(1+B)^3.$$

If $(1+B)u_p(z) \neq (1+A)$, *then* $(-4\kappa/c)u_p'(z) \in \mathcal{P}[A,B]$.

Corollary 2.4. *Let* $c \leq -1$, *and*

$$\kappa \geq \max\left\{\tfrac{c(c+1)}{2}, \tfrac{c}{2(c+1)}\right\}.$$

Then $(-4\kappa/c)(u_p(z)-1)$ *is close-to-convex of order* $(c+1)/c$ *with respect to the identity function.*

Corollary 2.5. *Let* c *be a nonzero real number, and* $\kappa \geq |c|/2$. *Then*

$$\mathrm{Re}(-4\kappa/c)u_p'(z) > 1/2.$$

3. Janowski starlikeness of generalized Bessel functions

This section contributes to find conditions to ensure a normalized and generalized Bessel functions $zu_p(z)$ in the class of Janowski starlike functions. For this purpose, first sufficient conditions for $u_p(z)$ to be Janowski convex is determined, and then an application of relation (3) yields conditions for $zu_p(z) \in \mathcal{S}^*[A,B]$.

Theorem 3.1. *Let* $c, \kappa \in \mathbb{R}$ *be such that* $(A-B)u_p'(z) \neq (1+B)zu_p''(z)$, $-1 \leq B \leq 0 < A \leq 1$. *Suppose*

$$\kappa(1+B) \geq \tfrac{(1+B)^2}{4(A-B)}|c| - (1+A-B). \tag{28}$$

Further let A, B, κ *and* c *satisfy*

$$(1+A-B+\kappa(1+B))(1-A+B+\kappa(1-B)) \tag{29}$$

$$\geq \tfrac{(1-B^2)^2}{16(A-B)^2}c^2 - \left|\tfrac{B-(A-B)(1+B^2)+(1-B^2)B\kappa}{2(A-B)}c\right| \tag{30}$$

If $0 \notin u_p'(\mathbb{D})$, $0 \notin u_p''(\mathbb{D})$, *then*

$$1 + \tfrac{zu_p''(z)}{u_p'(z)} \prec \tfrac{1+Az}{1+Bz}.$$

Proof. Define an analytic function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) := \tfrac{(A-B)u_p'(z)+(1-B)zu_p''(z)}{(A-B)u_p'(z)-(1+B)zu_p''(z)}.$$

Then

$$\tfrac{zu_p''(z)}{u_p'(z)} = \tfrac{(A-B)(p(z)-1)}{(p(z)+1)+B(p(z)-1)}, \tag{31}$$

and

$$\begin{aligned}\tfrac{z^2u_p'''(z)+zu_p''(z)}{zu_p''(z)} - \tfrac{zu_p''(z)}{u_p'(z)} &= \tfrac{zp'(z)}{(p(z)-1)} - \tfrac{(1+B)zp'(z)}{(p(z)+1)+B(p(z)-1)} \\ &= \tfrac{zp'(z)((p(z)+1)+B(p(z)-1)-(1+B)(p(z)-1))}{(p(z)-1)((p(z)+1)+B(p(z)-1))}.\end{aligned} \tag{32}$$

A rearrangement of (32) yields

$$\tfrac{zu_p'''(z)}{u_p''(z)} = \tfrac{2zp'(z)}{(p(z)-1)((p(z)+1)+B(p(z)-1))} - 1 + \tfrac{zu_p''(z)}{u_p'(z)}.$$

Thus,

$$\left(\frac{zu_p'''(z)}{u_p''(z)}\right)\left(\frac{zu_p''(z)}{u_p'(z)}\right)$$
$$= \frac{2(A-B)(p(z)-1)zp'(z)}{(p(z)-1)\Big((p(z)+1)+B(p(z)-1)\Big)^2} - \frac{(A-B)(p(z)-1)}{(p(z)+1)+B(p(z)-1)} + \frac{(A-B)^2(p(z)-1)^2}{((p(z)+1)+B(p(z)-1))^2}. \tag{33}$$

Now a differentiation of (2) leads to

$$4z^2u_p'''(z) + 4(\kappa+1)zu_p''(z) + czu_p'(z) = 0,$$

which give

$$\left(\frac{zu_p'''(z)}{u_p''(z)}\right)\left(\frac{zu_p''(z)}{u_p'(z)}\right) + (\kappa+1)\frac{zu_p''(z)}{u_p'(z)} + \frac{c}{4}z = 0. \tag{34}$$

Using (31) and (33), (34) yields

$$\frac{2(A-B)zp'(z)}{\Big((p(z)+1)+B(p(z)-1)\Big)^2} + \frac{(A-B)^2(p(z)-1)^2}{\Big((p(z)+1)+B(p(z)-1)\Big)^2} + \frac{(A-B)(p(z)-1)\kappa}{(p(z)+1)+B(p(z)-1)} + \frac{c}{4}z = 0,$$

equivalently

$$zp'(z) + \left(\frac{A-B}{2} + \frac{\kappa(1+B)}{2} + \frac{cz(1+B)^2}{8(A-B)}\right)(p(z))^2 - \left(A - B + \kappa B - \frac{c(1-B^2)}{4(A-B)}z\right)p(z)$$
$$+ \left(\frac{A-B}{2} - \frac{\kappa(1-B)}{2} + \frac{cz(1-B)^2}{8(A-B)}\right) = 0. \tag{35}$$

Define,

$$\Psi(p(z), zp'(z), z) := zp'(z) + F_1(p(z))^2 + F_2\, p(z) + F_3,$$

where

$$\begin{aligned} F_1 &= \frac{(A-B)}{2} + \frac{\kappa(1+B)}{2} + \frac{cz(1+B)^2}{8(A-B)}, \\ F_2 &= -(A-B) - \kappa B + \frac{c(1-B^2)}{4(A-B)}z, \\ F_3 &= \frac{(A-B)}{2} - \frac{\kappa(1-B)}{2} + \frac{cz(1-B)^2}{8(A-B)}. \end{aligned}$$

Thus, (35) yields $\Psi(p(z), zp'(z), z) \in \Omega = \{0\}$. Now with $z = x + iy \in \mathbb{D}$, let

$$\begin{aligned} G_1 := \operatorname{Re}(F_1) &= \frac{A-B}{2} + \frac{\kappa(1+B)}{2} + \frac{cx(1+B)^2}{8(A-B)} \\ &= \frac{1}{2}\left(A - B + \kappa(1+B) + \frac{cx(1+B)^2}{4(A-B)}\right), \\ G_2 := \operatorname{Re}(iF_2) &= -\frac{c(1-B^2)}{4(A-B)}y, \\ G_3 := \operatorname{Re}(F_3) &= \frac{A-B}{2} - \kappa\frac{1-B}{2} + \frac{cx(1-B)^2}{8(A-B)} \\ &= \frac{1}{2}\left(A - B - \kappa(1-B) + \frac{c(1-B)^2}{4(A-B)}x\right). \end{aligned}$$

For $\sigma \le -(1+\rho^2)/2$, $\rho \in \mathbb{R}$,

$$\begin{aligned} \operatorname{Re}\Psi(i\rho, \sigma, z) &= \sigma - G_1\rho^2 + G_2\rho + G_3 \\ &\le -\frac{1+2G_1}{2}\rho^2 + G_2\,\rho + \frac{2G_3+1}{2} := Q(\rho). \end{aligned}$$

Note that condition (28) implies $(1+2G_1)/2 > 0$. In this case, Q has a maximum at $\rho = G_2/(1+2G_1)$. Thus $Q(\rho) < 0$ for all real ρ provided

$$G_2^2 \leq (1+2G_1)(1-2G_3), \quad |x|, |y| < 1.$$

Since $y^2 < 1 - x^2$, it is left to show that

$$\begin{aligned}
&\tfrac{(1-B^2)^2}{16(A-B)^2} c^2 (1-x^2) \\
&\leq \left(1 + A - B + \kappa(1+B) + \tfrac{c(1+B)^2}{4(A-B)} x\right) (1 - A + B - \kappa(-1+B) \\
&\qquad - \tfrac{c(1-B)^2}{4(A-B)} x\right),
\end{aligned}$$

$|x| < 1$. The above inequality is equivalent to

$$H(x) := h_2(A,B)x + h_3(A,B) \geq 0, \tag{36}$$

where

$$\begin{aligned}
h_2(A,B) &= -\tfrac{(B-(A-B)(B^2+1)+(1-B^2)B\kappa)c}{2(A-B)}, \\
h_3(A,B) &= (1 + A - B + \kappa(1+B))(1 - A + B - \kappa(B-1)) - \tfrac{(1-B^2)^2}{16(A-B)^2} c^2.
\end{aligned}$$

Since $|x| < 1$, the left-hand side of the inequality (36) satisfy

$$h_2(A,B)x + h_3(A,B) \geq -|h_2(A,B)| + h_3(A,B).$$

Now it is evident from (29) that $H(x) \geq 0$ which establish the inequality (36).

Thus Ψ satisfies the hypothesis of Lemma 1.1, and hence Re $p(z) > 0$, or equivalently

$$\tfrac{(A-B)u_p' + (1-B)zu_p''}{(A-B)u_p' - (1+B)zu_p''} \prec \tfrac{1+z}{1-z}.$$

By definition of subordination, there exists an analytic self-map w of $\mathbb{D}$ with $w(0) = 0$ and

$$\tfrac{(A-B)u_p'(z) + (1-B)zu_p''(z)}{(A-B)u_p'(z) - (1+B)zu_p''(z)} = \tfrac{1+w(z)}{1-w(z)}.$$

A simple computation shows that

$$1 + \tfrac{zu_p''(z)}{u_p'(z)} = \tfrac{1+Aw(z)}{1+Bw(z)},$$

and hence

$$1 + \tfrac{zu_p''(z)}{u_p'(z)} \prec \tfrac{1+Az}{1+Bz}. \qquad \square$$

The relation (3) also shows that

$$\tfrac{z\,(zu_p(z)))'}{zu_p(z)} = 1 + \tfrac{zu_{p-1}''(z)}{u_{p-1}'(z)}.$$

Together with Theorem 3.1, it immediately yields the following result for $zu_p(z)) \in \mathcal{S}^*[A,B]$.

Theorem 3.2. *Let c and κ be real numbers such that $(A-B)u'_{p-1}(z) \neq (1+B)zu''_{p-1}(z)$, $-1 \leq B < A \leq 1$. Suppose*

$$\kappa(1+B) \geq \tfrac{(1+B)^2}{4(A-B)}\,|c| - (A-2B). \tag{37}$$

Further let A, B, κ and c satisfy

$$(A-2B+\kappa(1+B))(2B-A+\kappa(1-B)) \geq \tfrac{(1-B^2)^2}{16(A-B)}c^2 + \left|\tfrac{B^3-(A-B)(1+B^2)+(1-B^2)B\kappa}{2(A-B)}c\right|$$

Then $zu_p(z) \in \mathcal{S}^*[A,B]$.

References

[1] R. M. Ali, V. Ravichandran and N. Seenivasagan, Sufficient conditions for Janowski starlikeness, Int. J. Math. Math. Sci. **2007**, Art. ID 62925, 7 pp.

[2] R. M. Ali, R. Chandrashekar and V. Ravichandran, Janowski starlikeness for a class of analytic functions, Appl. Math. Lett. **24** (2011), no. 4, 501–505.

[3] Á. Baricz, Geometric properties of generalized Bessel functions, Publ. Math. Debrecen **73** (2008), no. 1-2, 155–178.

[4] Á. Baricz, Geometric properties of generalized Bessel functions of complex order, Mathematica **48(71)** (2006), no. 1, 13–18.

[5] Á. Baricz and S. Ponnusamy, Starlikeness and convexity of generalized Bessel functions, Integral Transforms Spec. Funct. **21** (2010), no. 9-10, 641–653.

[6] Á. Baricz and R. Szász, The radius of convexity of normalized Bessel functions of the first kind, Anal. Appl. (Singap.) **12** (2014), no. 5, 485–509.

[7] A. W. Goodman, *Univalent functions. Vol. I & II*, Mariner, Tampa, FL, 1983.

[8] W. Janowski, Some extremal problems for certain families of analytic functions. I, Ann. Polon. Math. **28** (1973), 297–326.

[9] C. P. McCarty, Functions with real part greater than α, Proc. Amer. Math. Soc. **35** (1972), 211–216.

[10] S. S. Miller and P. T. Mocanu, Univalence of Gaussian and confluent hypergeometric functions, Proc. Amer. Math. Soc. **110** (1990), no. 2, 333–342.

[11] S. S. Miller and P. T. Mocanu, Differential subordinations and inequalities in the complex plane, J. Differential Equations **67** (1987), no. 2, 199–211.

[12] S. S. Miller and P. T. Mocanu, *Differential subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Dekker, New York, 2000.

[13] S. Ponnusamy and M. Vuorinen, Univalence and convexity properties for confluent hypergeometric functions, Complex Variables Theory Appl. **36** (1998), no. 1, 73–97.

[14] St. Ruscheweyh and V. Singh, On the order of starlikeness of hypergeometric functions, J. Math. Anal. Appl. **113** (1986), no. 1, 1–11.

[15] V. Selinger, Geometric properties of normalized Bessel functions, Pure Math. Appl. **6** (1995), no. 2-3, 273–277.

[16] R. Szász and P. A. Kupán, About the univalence of the Bessel functions, Stud. Univ. Babeş-Bolyai Math. **54** (2009), no. 1, 127–132.

[17] N. M. Temme, *Special functions*, A Wiley-Interscience Publication, Wiley, New York, 1996.

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